Nonlinear Dyn
DOI 10.1007/s11071-015-2480-8

ORIGINAL PAPER

Nonlocal symmetry analysis and conservation laws to an third-order Burgers equation

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Received: 28 March 2015 / Accepted: 30 October 2015

Abstract The nonlocal residual symmetry related to truncated Painlevé expansion of third-order Burgers equation is performed. Then, the potential symmetries of the equation are derived; on the basis of nonlocal symmetries, we linearize the nonlinear third-order Burgers equation to a linear third-order PDE. Furthermore, the exact solutions of the potential equation are presented in terms of the symmetries. In particular, conservation laws are constructed of the equations.

Keywords Third-order Burger equation · Nonlocal symmetry analysis · Linearization · The multiplier approach · Conservation laws

Mathematics Subject Classification 22E70 · 70S10

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1 Introduction

It is known that the Burgers hierarchy is of the form [1–7]

$$u_t = K_m(u) = (D + u + u_x D^{-1})^{m-1} u_x, \quad m = 1, 2, \ldots \tag{1}$$

Substituting $n = 1, 2, 3, 4$ into (1), one can get few elements of the hierarchy (1), which are

$$u_t = u_x, \tag{2}$$

$$u_t = 2uu_x + u_{xx}, \tag{3}$$

$$u_t = 3uu_{xx} + u_{xxx} + 3u_x^2 + 3u^2 u_x, \tag{4}$$

$$u_t = u_{xxxx} + 10u_x u_{xx} + 4uu_{xxx} + 12uu_x^2 + 4u^3 u_x + 6u^2 u_{xx}. \tag{5}$$

They are of first order, second order, third order and fourth order, respectively. The third-order Burgers equation, similar to the second-order ones, appear in many physical and engineering fields [8–10], such as the plasma physics and fluid mechanics. In addition, these equations play a key role in nonlinear theory and mathematical physics, in particular in the integrable system, soliton theory, nonlinear wave theory, and so on. It is to be noted that Eq. (3) is the dissipative Burgers equation; by using the Hopf–Cole transformation, this equation can be reduced the heat equation $u_t - u_{xx} = 0$. Moreover, the third equation in Eq. (4) is the well-known Sharma–Tasso–Olver (STO) equation. The Burgers equation and the STO equation were investigated in many papers such as [1–19]. In paper

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[1], the authors considered the famous Burgres' equation and gave infinite series of flows. The authors in [4,5] presented exact solutions, which included self-similar solutions for Burgers hierarchy. Also, multiple kink solutions and multiple singular kink solutions of Burgers hierarchy were constructed in [6]. In Ref. [7], the authors considered the Painlev test, generalized symmetries, Bäklund transformations and exact solutions of the STO equation. In paper [18], the authors discussed group classification and exact solutions of generalized Burgers equations with linear damping and variable coefficients in detail. The authors gave the results about self-adjointness and conservation laws of a generalized Burgers equation in [19]. In paper [20], the authors considered the nonclassical potential symmetries of the Burgers equation. The authors analyzed the connection between symmetries and linearization in [21,22].

In this paper, we consider the nonlocal symmetries, exact solutions and derive the conservation laws of third-order Burgers equation (4) and its potential form. The remainder of this paper is organized as follows: In Sect. 2, nonlocal residual symmetry related to truncated Painlevé expansion of this equation is obtained. In Sect. 3, we employ potential symmetry on this equation and present all of the geometric vector fields are constructed. In Sect. 4, linearization by nonlocal symmetries is derived. In Sect. 5, similarity reductions and explicit solutions are presented. In Sect. 6, conservation laws are given. Finally, the main findings of the paper are recapitulated in Sect. 7.

2 Residual symmetry via Painlevé analysis

The STO equation (4) is Painlevé integrable [7,11]. In order to get residual symmetry via Painlevé analysis, we use the following truncated Painlevé expansion form

$$u=\frac{u_0}{\phi}+u_1. \tag{6}$$

Plugging (7) into (4), one can have

$$\begin{aligned}
&u_{1t}-3u_1u_{1xx}-u_{1xxx}-3u_{1x}^2-3u_1^2u_{1x}\\
&\quad+\phi^{(-1)}\Big(-6u_0u_1u_{1x}-3u_1^2u_{0x}-3u_0u_{1xx}\\
&\quad-3u_1u_{0xx}-6u_{0x}u_{1x}+u_{0t}-u_{0xxx}\Big)\\
&\quad+\phi^{(-2)}\Big(3\phi_xu_0u_1^2+6\phi_xu_0u_{1x}+6\phi_xu_1u_{0x}\\
&\quad+3\phi_{xx}u_1u_0-3u_0^2u_{1x}-6u_0u_1u_{0x}\\
&\quad-u_o\phi_t+3\phi_xu_{0xx}+3\phi_{xx}u_{0x}\\
&\quad+\phi_{xxx}u_0-3u_0u_{0xx}-3u_{0x}^2\Big)\\
&\quad+\phi^{(-3)}\Big(-6\phi_x^2u_0u_1\\
&\quad+6\phi_xu_0^2u_1-6\phi_x^2u_{0x}-6\phi_x\phi_{xx}u_0\\
&\quad+12\phi_xu_0u_{0x}+3\phi_{xx}u_0^2-3u_0^2u_{0x}\Big)\\
&\quad+\phi^{(-4)}\left(6\phi_x^3u_0-9\phi_x^2u_0^2+3\phi_xu_0^3\right).
\end{aligned} \tag{7}$$

Vanishing the coefficients of ϕ^{-4}, one can get

$$u_0=\phi_x, \tag{8}$$

or

$$u_0=2\phi_x. \tag{9}$$

The compatibility conditions at $j=1,3$ and this equation possesses the Painlevé property [7,11]. Moreover, from the coefficients of ϕ^0 and ϕ^1, one can obtain

Lemma 1 *$u_0=\phi_x$ is a symmetry of (4) with a solution of u_1.*

In order to proof this Lemma, we first give the definition of symmetry [12,13] and the Lemma 2.

Definition 1 [12,13] Given an evolution equation

$$u_t=K[u], \tag{10}$$

here $K[u]=K(x,t,u_x,u_{xx}\ldots)$, if $\sigma(x,t,u_x,u_{xx}\ldots)$ is a symmetry of Eq. (14), then $\sigma(x,t,u_x,u_{xx}\ldots)$ satisfy

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=K'\sigma, \tag{11}$$

where K' means the Fréchet derivative.

Lemma 2 *$\sigma=\sigma(x,t,u)$ is a symmetry of Eq. (4) if and only if*

$$\begin{aligned}
&\sigma_t-6\sigma uu_x-3u^2\sigma_x-3\sigma u_{xx}-3u\sigma_{xx}\\
&\quad-6\sigma_xu_x-\sigma_{xxx}=0,
\end{aligned} \tag{12}$$

where u is a solution of (4). This lemma is easily proved by Definition 1. From the coefficients of ϕ^0 and ϕ^1, we can get

$$\begin{aligned}
&u_{1t}-3u_1u_{1xx}-u_{1xxx}-3u_{1x}^2-3u_1^2u_{1x}=0,\\
&\Big(-6u_0u_1u_{1x}-3u_1^2u_{0x}\\
&\quad-3u_0u_{1xx}-3u_1u_{0xx}-6u_{0x}u_{1x}
\end{aligned}$$

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$$+u_{0t}-u_{0xxx})=0. \tag{13}$$

Substituting $u_0=\phi_x$ and u_1 into (16) and (4), one can get the Lemma 1.

It is noted that $u_0=\phi_x$ is a nonlocal symmetry with regard to Bäcklund transformation (6).

3 Point symmetry of the potential system and potential equation

In order to get more nonlocal symmetries, we use potential symmetry analysis to deal with (4). PDE (4) is equivalent to the potential system

$$\begin{aligned} &v_x=u,\\ &v_t=3uu_x+u_{xx}+u^3, \end{aligned} \tag{14}$$

and the potential equation

$$v_t=3v_xv_{xx}+v_{xxx}+(v_x)^3. \tag{15}$$

The point symmetry

$$\begin{aligned} &t^*=t+\varepsilon\tau(x,t,u,v)+O(\varepsilon^2),\\ &x^*=x+\varepsilon\xi(x,t,u,v)+O(\varepsilon^2),\\ &u^*=u+\varepsilon\eta(x,t,u,v)+O(\varepsilon^2),\\ &v^*=v+\varepsilon\psi(x,t,u,v)+O(\varepsilon^2), \end{aligned} \tag{16}$$

where ε is the group parameter, and

$$\begin{aligned} &X^{(3)}(v_x-u)=0,\\ &X^{(3)}(v_t-3uu_x-u_{xx}-u^3)=0, \end{aligned} \tag{17}$$

for any (u,v) that solves system (18);

$$\begin{aligned} X=&\tau(x,t,u,v)\frac{\partial}{\partial t}+\xi(x,t,u,v)\frac{\partial}{\partial x}\\ &+\eta(x,t,u,v)\frac{\partial}{\partial u}+\psi(x,t,u,v)\frac{\partial}{\partial v}, \end{aligned} \tag{18}$$

is the infinitesimal generator of the point symmetry (20);

$$\begin{aligned} X^{(3)}=&X+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}\\ &+\psi^x\frac{\partial}{\partial v_x}+\psi^t\frac{\partial}{\partial v_t}+\cdots \end{aligned} \tag{19}$$

with

$$\begin{aligned} &\eta^t=D_t(\eta)-u_xD_t(\xi)-u_tD_t(\tau),\\ &\eta^x=D_x(\eta)-u_xD_x(\xi)-u_tD_x(\tau),\\ &\psi^t=D_t(\psi)-v_xD_t(\xi)-v_tD_t(\tau),\\ &\psi^x=D_x(\psi)-v_xD_x(\xi)-v_tD_x(\tau),\\ &\quad\cdots \end{aligned} \tag{20}$$

Here, D_x and D_t denote the total derivative operator and are defined by

$$\begin{aligned} D_x=&\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+v_x\frac{\partial}{\partial v}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}\\ &+v_{xx}\frac{\partial}{\partial v_x}+v_{xt}\frac{\partial}{\partial v_t}+\cdots \end{aligned} \tag{21}$$

$$\begin{aligned} D_t=&\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+v_t\frac{\partial}{\partial v}+u_{xt}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}\\ &+v_{xt}\frac{\partial}{\partial v_x}+v_{tt}\frac{\partial}{\partial v_t}+\cdots \end{aligned} \tag{22}$$

Theorem 1 *The potential system (18) admits the point symmetry (20)*

$$\begin{aligned} &X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t},\\ &X_3=\frac{\partial}{\partial v},\quad X_4=\frac{\partial}{\partial t}+\frac{x}{3}\frac{\partial}{\partial x}-\frac{u}{3}\frac{\partial}{\partial u},\\ &X_F=\left(e^{-v}uF-e^{-v}F_x\right)\frac{\partial}{\partial u}-e^{-v}F\frac{\partial}{\partial v}, \end{aligned} \tag{23}$$

with

$$\begin{aligned} &\tau=c_1+c_2t,\quad \xi=\frac{c_2}{3}x+c_3,\\ &\eta=-\frac{1}{3}e^{-v}\left(e^{v}uc_2-3uF+3F_x\right),\\ &\psi=c_4-e^{-v}F, \end{aligned} \tag{24}$$

here $F=F(x,t)$, and F satisfies $F_t-F_{xxx}=0$, and hence, the scalar equation (4) admits the corresponding potential symmetry.

Proof: One can get the determining equations

$$\begin{aligned} &\tau_u=0,\\ &\tau_x=0,\\ &\tau_v=0,\\ &\psi_u=0,\\ &\xi_u=0,\\ &\xi_v-\eta_{uu}=0,\\ &\eta-u\psi_v+u^2\xi_v-\psi_x+u\xi_x=0,\\ &\psi_v+u\xi_v-\eta_u-\tau_t+2\xi_x=0,\\ &3\eta+\eta v-3u\psi_v+u\psi_{vv}+3u\eta_u+u\eta_{uv}\\ &\quad+3u\tau_t-u\tau_{tv}-3u\xi_x+2\eta_{xu}-\xi_{xx}=0,\\ &2u^2\eta+3u^2\eta v+u^2\eta_{vv}-\psi_t+u\xi_t+u^3\tau_t\\ &\quad+3u\eta_x+u^2\psi_x-u^3\xi_x+2u\eta_{xv}+\eta_{xx}=0. \end{aligned} \tag{25}$$

Solving these equations, one can obtain (27).

Theorem 2 *The potential equation* (19) *admits the point symmetry* (20)

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t}, X_3=\frac{\partial}{\partial v},$$
$$X_4=t\frac{\partial}{\partial t}+\frac{x}{3}\frac{\partial}{\partial x},\quad X_F=-e^{-v}F\frac{\partial}{\partial v}, \tag{26}$$

with

$$\tau=c_1+c_2t, \xi=\frac{c_2}{3}x+c_3, \psi=c_4-e^{-v}F, \tag{27}$$

here $F=F(x,t)$*, and* F *also satisfies* $F_t-F_{xxx}=0$.

Proof: One can get the determining equations

$$\xi_v=0,$$
$$\tau_v=0,$$
$$\tau_x=0,$$
$$\xi_{xvv}=0,$$
$$\psi_v+\psi_{vv}=0,$$
$$\tau_t-3\xi_x=0,$$
$$\psi_t-\psi_{xxx}=0,$$
$$\xi_t+2\xi_{xxx}=0,$$
$$\psi_x+\psi_{xv}-\xi_{xx}=0. \tag{28}$$

Solving these equations, one can obtain (30).

Notes

1. In the above, the vector fields X_F are not recoverable point symmetries of the original equation and are potential symmetries (nonlocal).
2. The conservation laws used to construct the potential system are not unique if the there exists more than such conserved forms. Moreover, not all the potential forms may produce potential symmetries. If the respective conserved form is physical, it is purely coincidental.

4 Linearization (4) using nonlocal symmetries

One of the approaches that involve invariance deal with derivative or integral dependent coefficients in the vector fields/symmetries. The latter is somewhat complicated but may be constructed via the conserved form of the differential equation often referred to as potential form. The symmetries of these systems are nonlocal/potential symmetries of the original equation (see [21–24]).

In order to linearize (4), we use the Theorems as shown in [21,22]. In the previous section, we showed that the system (17)

$$v_x=u,$$
$$v_t=3uu_x+u_{xx}+u^3, \tag{29}$$

admits the infinite-parameter Lie group of point transformations with infinitesimal generator

$$X_F=e^{-v}\left\{\left(uF^1-F^2\right)\frac{\partial}{\partial u}-F^1\frac{\partial}{\partial v}\right\}, \tag{30}$$

where $F=(F^1,F^2)$ is an arbitrary solution of the linear system of equations

$$\frac{\partial F^1}{\partial x}=F^2,$$
$$\frac{\partial F^2}{\partial x^2}=\frac{\partial F^1}{\partial t}. \tag{31}$$

Then, one has

$$\tau=0,\quad \xi=0,\quad \eta=e^{-v}\left(uF^1-F^2\right),$$
$$\psi=e^{-v}F^1. \tag{32}$$

We now identify

$$\alpha_1^1=\alpha_1^2=\alpha_2^1=\beta_2^2=0,\quad \beta_1^1=ue^{-v},$$
$$\beta_1^2=-e^{-v},\quad \beta_2^1=-e^{-v}. \tag{33}$$

In this way, one can get

$$ue^{-v}\frac{\partial\Psi^1}{\partial u}-e^{-v}\frac{\partial\Psi^1}{\partial v}=1,\quad ue^{-v}\frac{\partial\Psi^2}{\partial u}\quad-e^{-v}\frac{\partial\Psi^2}{\partial v}=0,$$
$$-e^{-v}\frac{\partial\Psi^1}{\partial u}=0,\quad -e^{-v}\frac{\partial\Psi^2}{\partial u}=1. \tag{34}$$

It is easy to get a particular solution, which is

$$\Psi^1=-e^v,\quad \Psi^2=-ue^v. \tag{35}$$

Then, the invertible mapping μ is given by

$$z_1=x,\quad z_2=t,\quad w^1=-e^v,\quad w^2=-ue^v, \tag{36}$$

which transforms (29) to the following linear system

$$\frac{\partial w^1}{\partial z_1}=w^2,$$
$$\frac{\partial^2 w^2}{\partial z_1^2}=\frac{\partial w^1}{\partial z_2}. \tag{37}$$

Consequently, the noninvertible mapping

$$u=\frac{w^2}{w^1}=\frac{\frac{\partial w^1}{\partial z_1}}{w^1}, \tag{38}$$

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embeds third-order Burgers equation (4) in the following linear equation

$$\frac{\partial^3 w^1}{\partial z_1^3} = \frac{\partial w^1}{\partial z_2}. \tag{39}$$

For brevity, we rewrite the (39) as follows

$$u_t = u_{xxx}. \tag{40}$$

It should be noted that the mapping (38) is the famous Hopf–Cole transformation. Also, we note that this result is the same as in [4,7]. It should be stressed that, however, we analyzed the results from another perspective.

5 Reduction equations and similarity solutions related to potential equation

In this equation, we deal with the invariant solutions of the potential equation (19). Now we consider the following cases

Case 1 X_1.

For this case, the invariants is $\xi = t$, and we get the trivial solution $v = c_1$.

Case 2 X_2.

For this case, the invariants is $\xi = x$, and one can obtain

$$3f'f'' + f''' + (f')^3 = 0. \tag{41}$$

For this case, we get the integrating factor

$$\Lambda = C_1 \mathrm{e}^{2f} + \frac{1}{2}\mathrm{e}^f \left(C_2\xi^2 + 2C_3\xi + 2C_4\right), \tag{42}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

In this way, (41) can be reduced into following cases:

$$\frac{1}{2}\mathrm{e}^{2f}\left(f'^2 + 2f''\right) = c_1. \tag{43}$$

$$\mathrm{e}^f\left(f'^2 + f''\right) = c_2. \tag{44}$$

$$\mathrm{e}^f\left(\xi f'^2 + \xi f'' - f'\right) = c_3. \tag{45}$$

$$\frac{1}{2}\mathrm{e}^f\left(\xi^2 f'^2 + \xi^2 f'' - 2\xi f' + 2\right) = c_4. \tag{46}$$

Solving the first one, one can obtain

$$f = \ln\left(\frac{{C_1}^2{C_2}^2 + 2\,{C_1}^2 C_2\xi + {C_1}^2\xi^2 + 8\,c_1}{4C_1}\right). \tag{47}$$

Putting (47) into (44), (45), (46), one can get

$$c_2 = \frac{C_1}{2}, \quad c_3 = -\frac{C_1C_2}{2}, \quad c_4 = \frac{(C_1C_2)^2 + 8c_1}{2C_1}. \tag{48}$$

Case 3 $\lambda X_1 + X_2$.

For the linear combination, the invariants is $\xi = x - \lambda t$ and $f(\xi)$, and one can have

$$\lambda f' + 3f'f'' + f''' + (f')^3 = 0. \tag{49}$$

For this case, we get the integrating factor

$$\Lambda = C_1\mathrm{e}^{2f} + \left(C_2 + C_3\sin\left(\sqrt{\lambda}\xi\right) + C_4\cos\left(\sqrt{\lambda}\xi\right)\right)\mathrm{e}^f, \tag{50}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

In this way, one can have

$$\frac{1}{2}\mathrm{e}^{2f}\left(f'^2 + 2f'' + \lambda\right) = c_1. \tag{51}$$

$$\mathrm{e}^f\left(f'^2 + f'' + \lambda\right) = c_2. \tag{52}$$

$$-\mathrm{e}^f\left(\cos(\sqrt{\lambda}\xi)\sqrt{\lambda}f' - \sin(\sqrt{\lambda}\xi)f'^2 - \sin(\sqrt{\lambda}\xi)f''\right) = c_3. \tag{53}$$

$$\mathrm{e}^f\left(\cos(\sqrt{\lambda}\xi)f'^2 + \sqrt{\lambda}\sin(\sqrt{\lambda}\xi)f' + \cos(\sqrt{\lambda}\xi)f''\right) = c_4. \tag{54}$$

Solving (51), one can get

$$f = \ln\left(-\frac{\sin\left(\sqrt{\lambda}x\right)\sqrt{\lambda}C_1 - C_2\cos\left(\sqrt{\lambda}x\right)\sqrt{\lambda} - c_2}{\lambda}\right). \tag{55}$$

Substituting (55) into (52), (53) and (54), one can get

$$c_1 = -\frac{{C_1}^2\lambda + {C_2}^2\lambda - {c_2}^2}{2\lambda}, \quad c_3 = C_1\sqrt{\lambda}, \quad c_4 = -C_2\sqrt{\lambda}. \tag{56}$$

Case 4 X_4.

One can get the invariants and function are $\xi = xt^{-\frac{1}{3}}$ and $f(\xi)$, and one can have

$$\frac{1}{3}\xi f' + 3f'f'' + f''' + (f')^3 = 0. \tag{57}$$

We get the integrating factor

$$\Lambda = \mathrm{e}^f\left(C_1\xi^2 \mathit{hypergeom}\left([1], \left[\frac{4}{3}, \frac{5}{3}\right], -\frac{\xi^3}{27}\right)\right.$$

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$$+C_2 I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right)(-\xi^3)^{\frac{1}{6}}$$

$$+C_3 \xi I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right)(-\xi^3)^{-\frac{1}{6}}\Bigg). \quad (58)$$

Therefore, one can obtain

$$\frac{e^f}{1680}\Bigg\{3\xi^6 hy\left([3], \left[\frac{10}{3}, \frac{11}{3}\right], -\frac{\xi^3}{27}\right)$$

$$+84\xi^4 hy\left([2], \left[\frac{7}{3}, \frac{8}{3}\right], -\frac{\xi^3}{27}\right) f'$$

$$+1680\xi^2 hy\left([1], \left[\frac{4}{3}, \frac{5}{3}\right], -\frac{\xi^3}{27}\right) f'^2$$

$$+1680\xi^2 hy\left([1], \left[\frac{4}{3}, \frac{5}{3}\right], -\frac{\xi^3}{27}\right) f''$$

$$+560\xi^3 hy\left([1], \left[\frac{4}{3}, \frac{5}{3}\right], -\frac{\xi^3}{27}\right)$$

$$-504\xi^3 hy\left([2], \left[\frac{7}{3}, \frac{8}{3}\right], -\frac{\xi^3}{27}\right)$$

$$-3360\xi hy\left([1], \left[\frac{4}{3}, \frac{5}{3}\right], -\frac{\xi^3}{27}\right) f'$$

$$+3360 hy\left([1], \left[\frac{4}{3}, \frac{5}{3}\right], -\frac{\xi^3}{27}\right)\Bigg\} = c_1. \quad (59)$$

$$\frac{\sqrt{3}}{3\xi} e^f (-\xi^3)^{\frac{1}{6}}\Bigg(\sqrt{3}\xi I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right) f'^2$$

$$-\sqrt{-\xi^3} I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right) f'$$

$$+\sqrt{3}\xi I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right) f''\Bigg) = c_2. \quad (60)$$

$$\frac{\sqrt{3}}{3\xi^2} e^f (-\xi^3)^{\frac{1}{3}}\Bigg(\sqrt{3} I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right) f'^2 \sqrt{-\xi^3}$$

$$+I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right) f'\xi^2$$

$$+\sqrt{3} I\left(\frac{-1}{3}, \frac{2\sqrt{3}\sqrt{-\xi^3}}{9}\right) f''\sqrt{-\xi^3}\Bigg) = c_3. \quad (61)$$

Solving (60), one can get

$$f = \ln\Bigg(\int e^{-\frac{1}{3}\sqrt{3}\int \sqrt{-\xi^3} I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1}\xi^{-1} d\xi}$$

$$\times \frac{1}{\sqrt[6]{-\xi^3}}\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1} d\xi$$

$$\times \int e^{\frac{1}{3}\sqrt{3}\int \sqrt{-\xi^3} I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1}\xi^{-1} d\xi} d\xi c_2$$

$$-\int e^{\frac{1}{3}\sqrt{3}\int \sqrt{-\xi^3} I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1}\xi^{-1} d\xi} d\xi C_1$$

$$-c_2 \int\int e^{\frac{1}{3}\sqrt{3}\int \sqrt{-\xi^3} I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1}\xi^{-1} d\xi} d\xi$$

$$e^{-\frac{1}{3}\sqrt{3}\int \sqrt{-\xi^3} I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1}\xi^{-1} d\xi}$$

$$\times \frac{1}{\sqrt[6]{-\xi^3}}\left(6\frac{I\left(\frac{2}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)}{\sqrt{-3\xi^3}} + I\left(\frac{5}{3}, \frac{2}{9}\sqrt{-3\xi^3}\right)\right)^{-1} d\xi + C_2\Bigg), \quad (62)$$

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where I is the Bessel functions of the first kind and hy is the *hypergeom* functions.

6 Conservation law

In this section, using the multipliers method [23,24], we will handle the conservation law of the (4), (40) and (15).

6.1 Basic concepts

Given a system of r PDEs of k-order with x and u [22–25]

$$F_\iota(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0,\quad \iota = 1, 2, \ldots, r, \tag{63}$$

where $u_{(1)} = \{u_i^\iota\}$, $u_{(2)} = \{u_{ij}^\iota\}, \ldots$, and $u_i^\iota = \frac{\partial u^\iota}{\partial x_i}$, $u_{ij}^\iota = \frac{\partial^2 u^\iota}{\partial x_i \partial x_j}, \ldots$.

1. In this paper, the total derivative operators D_i are defined by (21) and (22).
2. Multipliers for PDE system (63) are undetermined functions $\{\Lambda^\alpha[U]\}$, which satisfies the following equation
$$\Lambda^\iota[U]F_\iota[U] = D_i T^i[U], \tag{64}$$
for a set of functions $\{T^i[U]\}$ [22–24].
If $U^\sigma = u^\sigma(x)$ is a solution of PDE system, then one can get the conservation law
$$D_i T^i[U] = 0, \tag{65}$$
of PDE system (63), where $T^i[U]$ are the conserved densities.
3. The Euler operators defined by [22–25]
$$E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u_i^j} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \cdots i_s}} + \cdots, \tag{66}$$
for each $j = 1, 2, \ldots, m$. By employing the following equation,
$$E_{U^j}(\Lambda^\iota[U]F_\iota[U]) \equiv 0,\quad j = 1, \ldots, N. \tag{67}$$

one can get a set of multipliers.

6.2 Conservation laws of (4)

From the above, we suppose the conservation law is given by $D_x T^x + D_t T^t = 0$ on the solutions of (4). For the fourth-order multiplier, one can get

$$\Lambda_t = 0,\quad \Lambda_x = 0,\quad \Lambda_u = 0,\quad \Lambda_{u_x} = 0,$$
$$\Lambda_{u_{xx}} = 0,\quad \Lambda_{u_{xxx}} = 0,\quad \Lambda_{u_{xxxx}} = 0. \tag{68}$$

Solving them, one can get

$$\Lambda = C_1. \tag{69}$$

Therefore, one can yield the fluxes

$$T^t = u,$$
$$T^x = -u^3 - 3uu_x - u_{xx}. \tag{70}$$

6.3 Conservation laws of (40)

Applying the previous steps, for the fourth-order multiplier, one can get

$$\Lambda_1 = \frac{9t^2 u_{xxxx}}{2} + 3xtu_{xx} + 3tu_x + \frac{x^2 u}{2},$$
$$\Lambda_2 = xu + 3tu_{xx},$$
$$\Lambda_3 = u,$$
$$\Lambda_4 = xu_{xx} + 3tu_{xxxx} + u_x,$$
$$\Lambda_5 = u_{xx},$$
$$\Lambda_6 = u_{xxxx}, \tag{71}$$

for which the respective conserved quantities are

$$T_1^t = \frac{u(u_x^2 + 6u_{xxtx} + 9t^2 u_{xxxx} + 6u_{xt})}{4},$$
$$T_1^x = -\frac{xuu_x}{2} - \frac{9tuu_{xxx}}{2} - \frac{x^2 uu_{xx}}{2} + \frac{3tu_x u_{xx}}{2} - \frac{3xtu_{xx}^2}{2} - \frac{9t^2 uu_{txxx}}{4} + \frac{9t^2 u_x u_{txx}}{4} - \frac{9t^2 u_{xx} u_{tx}}{4} + \frac{9t^2 u_t u_{xxx}}{4} - \frac{u^2}{2} + \frac{x^2 u_x^2}{4} - \frac{9t^2 u_{xxx}^2}{4} + \frac{3xtu_x u_t}{2} - \frac{3xtuu_{tx}}{2}. \tag{72}$$

$$T_2^t = \frac{u(xu + 3tu_{xx})}{2},$$
$$T_2^x = -xuu_{xx} - \frac{uu_x}{2} + \frac{xu_x^2}{2} - \frac{3tu_{xx}^2}{2} - \frac{3tuu_{tx}}{2} + \frac{3tu_t u_x}{2}. \tag{73}$$

$$T_3^t = \frac{u^2}{2},$$
$$T_3^x = -uu_{xx} + \frac{u_x}{2}. \tag{74}$$

$$T_4^t = \frac{u(xu_{xx} + 3tu_{xxxx} + u_x)}{2},$$
$$T_4^x = \frac{xu_x u_t}{2} + \frac{3tu_{xxx} u_t}{2} - \frac{xuu_{tx}}{2} - \frac{3tuu_{txxx}}{2}$$

$$+\frac{3tu_xu_{txx}}{2}-\frac{3tu_{tx}u_{xx}}{2}-\frac{3uu_{xxx}}{2}$$
$$-\frac{3tu_{xxx}^2}{2}+\frac{u_xu_{xx}}{2}-\frac{xu_{xx}^2}{2}. \quad (75)$$

$$T_5^t=\frac{uu_{xx}}{2},$$
$$T_5^x=-\frac{u_{xx}^2}{2}-\frac{uu_{tx}}{2}+\frac{u_xu_t}{2}. \quad (76)$$

$$T_6^t=\frac{uu_{xxxx}}{2},$$
$$T_6^x=-\frac{uu_{txxx}}{2}+\frac{u_xu_{txx}}{2}-\frac{u_{xx}u_{tx}}{2}$$
$$+\frac{u_{xxx}u_t}{2}-\frac{u_{xxx}^2}{2}. \quad (77)$$

6.4 Conservation laws of (15)

Once again applying the previous steps, for the fourth-order multiplier, one can get

$$\Lambda_1=\frac{e^{2u}}{18}\big(9t^2u_x^4+54t^2u_x^2u_{xx}+6txu_x^2$$
$$+36t^2u_xu_{xxx}+27t^2u_{xx}^2$$
$$+6txu_{xx}+9t^2u_{xxxx}+6tu_x+x^2\big),$$
$$\Lambda_2=\frac{e^{2u}}{3}\big(3tu_x^4+18tu_x^2u_{xx}+xu_x^2+12tu_xu_{xxx}$$
$$+9tu_{xx}^2+xu_{xx}+3tu_{xxxx}+u_x\big),$$
$$\Lambda_3=e^{2u}\left(u_x^4+6u_x^2u_{xx}+4u_xu_{xxx}+3u_{xx}^2+u_{xxxx}\right),$$
$$\Lambda_4=\frac{e^{2u}}{3}\big(3tu_x^2+3tu_{xx}+x\big),$$
$$\Lambda_5=e^{2u}(u_x^2+u_{xx}),$$
$$\Lambda_6=e^{2u}, \quad (78)$$

for which the respective conserved quantities are given in Appendix.

7 Conclusions

In the present paper, we studied the nonlocal symmetries, similarity reduction and explicit solutions of the third-order Burgers equation. In particular, we linearized the equation to the third-order linear PDE via symmetries. Also, some explicit solutions of potential equation are constructed. Furthermore, we also give the conservation laws of the third-order Burgers equation, the reduced equation and the potential equation. In the future, we will deal with the high-order equation, such as the fourth-order Burgers equation, even more high-order equation and then with variable coefficients.

Acknowledgments This work was supported by International Graduate Exchange Program of Beijing Institute of Technology (1320012341502), Research Project of China Scholarship Council (No. 201406030057), National Natural Science Foundation of China (NNSFC) (Grant No. 11171022), Graduate Student Science and Technology Innovation Activities of Beijing Institute of Technology (No. 2014cx10037). The authors thank the reviewers for the useful comments and suggestions which have been incorporated into the text.

Appendix

$$T_1^t=\frac{27t^2u_x^4e^{2u}+2x^2u^4e^{2u}+6tu^3u_x-27t^2u^2u_{xx}^2+9t^2u^3u_{xxxx}-2x^2u^4}{72u^4}$$
$$+\frac{27t^2e^{2u}u^2u_{xx}^2-27t^2u_x^4-9t^2e^{2u}u^3u_{xxxx}+12te^{2u}u_xu^4+54t^2e^{2u}u^4u_{xx}^2}{72u^4}$$
$$+\frac{18t^2e^{2u}u^4u_{xxxx}-54t^2e^{2u}u^3u_{xx}^2+18t^2e^{2u}u^4u_x^4-36t^2e^{2u}u^3u_x^4+54t^2e^{2u}u^2u_x^4}{72u^4}$$
$$+\frac{6tu^3u_{xxx}-54t^2e^{2u}uu_x^4-36t^2u^2u_xu_{xxx}+81t^2uu_x^2u_{xx}-6txu^2u_x^2-6te^{2u}u^3u_x}{72u^4}$$
$$+\frac{72t^2e^{2u}u^4u_xu_{xxx}+108t^2e^{2u}u^4u_x^2u_{xx}-162t^2e^{2u}u^3u_x^2u_{xx}+12txe^{2u}u^4u_x^2}{72u^4}$$
$$+\frac{12txe^{2u}u^4u_{xx}-72t^2e^{2u}u^3u_xu_{xxx}+162t^2e^{2u}u^2u_x^2u_{xx}-12xte^{2u}u^3u_x^2}{72u^4}$$
$$+\frac{36t^2e^{2u}u^2u_xu_{xxx}-6txe^{2u}u^3u_{xx}-81t^2e^{2u}uu_{xx}u_x^2+6txe^{2u}u^2u_x^2}{72u^4},$$

$$
\begin{aligned}
T_1^x = {} & \frac{4\mathrm{e}^{2u}u^4 - 4u^4 + 27t^2\mathrm{e}^{2u}u^2u_xu_{txx} + 27t^2\mathrm{e}^{2u}u^2u_{xx}u_{tx} + 54t\mathrm{e}^{2u}u^2u_xu_{xx}}{-72u^4} \\
& + \frac{12tx\mathrm{e}^{2u}u^4u_{xx}^2 + 96t\mathrm{e}^{2u}u^4u_xu_{xx} - 108t\mathrm{e}^{2u}u^3u_xu_{xx} + 36t^2\mathrm{e}^{2u}u^4u_xu_{txx}}{-72u^4} \\
& + \frac{108t^2\mathrm{e}^{2u}u^4u_x^4u_{xx} - 18t^2\mathrm{e}^{2u}u^3u_tu_{xxx} - 54t^2\mathrm{e}^{2u}u^3u_xu_{txx} - 54t^2\mathrm{e}^{2u}u^3u_{tx}u_{xx}}{-72u^4} \\
& + \frac{12tx\mathrm{e}^{2u}u^4u_x^4 + 162t^2\mathrm{e}^{2u}u^4u_x^2u_{xx}^2 + 36t^2\mathrm{e}^{2u}u^4u_x^3u_{xxx} - 54t^2\mathrm{e}^{2u}u^3u_x^2u_{tx}}{-72u^4} \\
& + \frac{54t^2\mathrm{e}^{2u}u^2u_{tx}u_x^2 + 72t^2\mathrm{e}^{2u}u^4u_{tx}u_{xx} + 12tx\mathrm{e}^{2u}u^4u_{tx} - 36t^2\mathrm{e}^{2u}u^3u_tu_x^3}{-72u^4} \\
& + \frac{54t^2\mathrm{e}^{2u}u^2u_tu_x^3 - 54t^2\mathrm{e}^{2u}u_t^uu_x^3 + 54t^2uu_tu_xu_{xx} - 6xtu^2u_tu_x}{-72u^4} \\
& + \frac{9t^2\mathrm{e}^{2u}u^2u_tu_{xxx} - 27t^2\mathrm{e}^{2u}uu_{tx}u_x^2 - 6tx\mathrm{e}^{2u}u^3u_{tx} + 4x^2\mathrm{e}^{2u}u^4u_{xx}}{-72u^4} \\
& + \frac{6txu^3u_{tx} - 9t^2u^2u_tu_{xxx} - 27t^2u^2u_{tx}u_{xx} - 27t^2u^2u_xu_{txx} - 54tu^2u_xu_{xx}}{-72u^4} \\
& + \frac{27t^2uu_{tx}u_x^2 + 36t\mathrm{e}^{2u}u^4u_{xxx} + 24t\mathrm{e}^{2u}u^4u_x^3 + 18t^2\mathrm{e}^{2u}u^4u_x^6 + 27t^2\mathrm{e}^{2u}u_tu_x^3}{-72u^4} \\
& + \frac{2x^2\mathrm{e}^{2u}u^4u_x^2 - 36t\mathrm{e}^{2u}u^3u_x^3 + 18t^2\mathrm{e}^{2u}u^4u_{xxx}^2 + 36t\mathrm{e}^{2u}u^2u_x^3 - 6x\mathrm{e}^{2u}u^3u_x}{-72u^4} \\
& + \frac{4x\mathrm{e}^{2u}u^4u_x - 18t\mathrm{e}^{2u}uu_x^3 - 18t\mathrm{e}^{2u}u^3u_{xxx} - 9t^2\mathrm{e}^{2u}u^3u_{txxx} + 18t^2\mathrm{e}^{2u}u^4u_{txxx}}{-72u^4} \\
& + \frac{18tu^3u_{xxx} + 9t^2u^3u_{txxx} + 6xu^3u_x - 27t^2u_tu_x^3 + 108t^2\mathrm{e}^{2u}u^4u_xu_{xx}u_{xxx}}{-72u^4} \\
& + \frac{18tuu_x^3 - 54t^2\mathrm{e}^{2u}uu_tu_xu_{xx} + 6tx\mathrm{e}^{2u}u^2u_tu_x - 108t^2\mathrm{e}^{2u}u^3u_tu_xu_{xx}}{-72u^4} \\
& + \frac{108t^2\mathrm{e}^{2u}u^2u_tu_xu_{xx} - 12tx\mathrm{e}^{2u}u^3u_tu_x + 24tx\mathrm{e}^{2u}u^4u_x^2u_{xx}}{-72u^4}.
\end{aligned}
$$

$$
\begin{aligned}
T_2^t = {} & \frac{-3t\mathrm{e}^{2u}u^3u_{xxxx} + x\mathrm{e}^{2u}u^2u_x^2u_{xx} + 9t\mathrm{e}^{2u}u^2u_{xx}^2 - x\mathrm{e}^{2u}u^3u_{xx} + 2x\mathrm{e}^{2u}u^4u_x^2}{12u^4} \\
& + \frac{24t\mathrm{e}^{2u}u^4u_xu_{xxx} + 36t\mathrm{e}^{2u}u^4u_x^2u_{xx} - 54t\mathrm{e}^{2u}u^3u_x^2u_{xx} - 24t\mathrm{e}^{2u}u^3u_xu_{xxx}}{12u^4} \\
& + \frac{-27t\mathrm{e}^{2u}uu_{xx}u_x^2 + 12t\mathrm{e}^{2u}u^2u_xu_{xxx} + u^3u_x + 18t\mathrm{e}^{2u}u^4u_{xx}^2 - 18t\mathrm{e}^{2u}u^3u_{xx}^2}{12u^4} \\
& + \frac{6t\mathrm{e}^{2u}u^4u_x^4 - 12t\mathrm{e}^{2u}u^3u_x^4 + 18t\mathrm{e}^{2u}u^2u_x^4 - 18t\mathrm{e}^{2u}uu_x^4 + 6t\mathrm{e}^{2u}u^4u_{xxxx}}{12u^4} \\
& + \frac{-12tu^2u_xu_{xxx} - 9tu_x^4 + 2\mathrm{e}^{2u}u^4u_x + 9t\mathrm{e}^{2u}u_x^4 - \mathrm{e}^{2u}u^3u_x + 3tu^3u_{xxxx} - xu^2u_x^2}{12u^4} \\
& + \frac{54t\mathrm{e}^{2u}u^2u_x^2u_{xx} + 2x\mathrm{e}^{2u}u^4u_{xx} + 27tuu_x^2u_{xx} - 9tu^2u_{xx}^2 + xu^3u_{xx}}{12u^4},
\end{aligned}
$$

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$$T_2^x = \frac{36te^{2u}u^4u_xu_{xx}u_{xxx} - 18te^{2u}uu_tu_xu_{xx} - 36te^{2u}u^3u_tu_xu_{xx} + 36te^{2u}u^2u_tu_xu_{xx}}{-12u^4}$$

$$+ \frac{3u^3u_{xxx} + 3uu_x^3 - xu_tu_xu^2 - 3tu^2u_tu_{xxx} - 9tu^2u_{tx}u_{xx} - 9tu^2u_{txx}u_x}{-12u^4}$$

$$+ \frac{2xe^{2u}u^4u_x^4 + 6tu^4e^{2u}u_x^6 + 9te^{2u}u_tu_x^3 + 2xe^{2u}u^4u_{xx}^2 + 16e^{2u}u^4u_xu_{xx}}{-12u^4}$$

$$+ \frac{-xe^{2u}u^3u_{tx} + 9e^{2u}u^2u_xu_{xx} - 3te^{2u}u^3u_{txxx} - 18e^{2u}u^3u_xu_{xx} + 6te^{2u}u^4u_{txxx}}{-12u^4}$$

$$+ \frac{2xe^{2u}u^4u_{tx} + 24te^{2u}u^4u_{tx}u_{xx} + 12te^{2u}u^4u_xu_{txx} - 6te^{2u}u_tu^3u_{xxx} - 2xe^{2u}u^3u_xu_t}{-12u^4}$$

$$+ \frac{54te^{2u}u^4u_x^2u_{xx}^2 - 18te^{2u}u^3u_{txx}u_x - 18te^{2u}u^3u_{tx}u_{xx} + 12te^{2u}u^4u_x^3u_{xxx}}{-12u^4}$$

$$+ \frac{18te^{2u}u^2u_x^2u_{tx} - 18te^{2u}u^3u_x^2u_{tx} + 4xe^{2u}u^4u_{xx}u_x^2 - 12te^{2u}u^3u_tu_x^3}{-12u^4}$$

$$+ \frac{18te^{2u}u^2u_tu_x^3 - 18te^{2u}uu_tu_x^3 + 18tuu_tu_xu_{xx} - 9te^{2u}uu_{tx}u_x^2 + xe^{2u}u^2u_tu_x}{-12u^4}$$

$$+ \frac{9te^{2u}u^2u_xu_{txx} + 9te^{2u}u^2u_{xx}u_{tx} + 3te^{2u}u^2u_tu_{xxx} - 3e^{2u}u^3u_{xxx} - 3e^{2u}uu_x^3}{-12u^4}$$

$$+ \frac{6e^{2u}u^4u_{xxx} - 6e^{2u}u^3u_x^3 + 6e^{2u}u^2u_x^3 + 4e^{2u}u^4u_x^3 + 12te^{2u}u^4u_x^2u_{tx} + 6te^{2u}u^4u_{xxx}^2}{-12u^4}$$

$$+ \frac{36te^{2u}u^4u_x^4u_{xx}3tu^3u_{txxx} - 9u^2u_xu_{xx} + xu^3u_{tx} - 9tu_tu_x^3 + 9tuu_{tx}u_x^2}{-12u^4}.$$

$$T_3^t = \frac{2e^{2u}u^4u_x^4 + 12e^{2u}u^4u_x^2u_{xx} - 4e^{2u}u^3u_x^4 + 8e^{2u}u^4u_xu_{xxx} + 6e^{2u}u^4u_{xx}^2}{4u^4}$$

$$+ \frac{-18e^{2u}u^3u_x^2u_{xx} + 6e^{2u}u^2u_x^4 + 2e^{2u}u^4u_{xxxx} - 8e^{2u}u^3u_xu_{xxx} - 6e^{2u}u^3u_{xx}^2}{4u^4},$$

$$+ \frac{18e^{2u}u^2u_x^2u_{xx} - 6e^{2u}uu_x^4 - e^{2u}u^3u_{xxxx} + 4e^{2u}u^2u_xu_{xxx} + 3e^{2u}u^2u_{xx}^2}{4u^4},$$

$$+ \frac{9e^{2u}uu_x^2u_{xx} + 3e^{2u}uu_x^4 + u^3u_{xxxx} - 4u^2u_xu_{xxx} - 3u^2u_{xx}^2 + 9uu_x^2u_{xx} - 3u_x^4}{4u^4},$$

$$
\begin{aligned}
T_3^x &= \frac{-3u_t u_x^3 + u^3 u_{txxx} - 4\mathrm{e}^{2u} u^3 u_t u_x^3 + 4\mathrm{e}^{2u} u^4 u_{tx} u_x^2 + 12\mathrm{e}^{2u} u^4 u_x^4 u_{xx} + 18\mathrm{e}^{2u} u^4 u_x^2 u_{xx}^2}{4u^4} \\
&+ \frac{-6\mathrm{e}^{2u} u_x u_t u u_{xx} - 12u^3 \mathrm{e}^{2u} u_x u_t u_{xx} + 12u^2 \mathrm{e}^{2u} u_x u_t u_{xx} + 12\mathrm{e}^{2u} u^4 u_x u_{xx} u_{xxx}}{4u^4} \\
&+ \frac{3uu_{tx} u_x^2 - \mathrm{e}^{2u} u^3 u_{txxx} + 3\mathrm{e}^{2u} u_t u_x^3 + 2\mathrm{e}^{2u} u^4 u_{xxx}^2 + 2\mathrm{e}^{2u} u^4 u_{txxx} + 2\mathrm{e}^{2u} u^4 u_x^6}{4u^4} \\
&+ \frac{-3u^2 u_{txx} u_x - 3u^2 u_{tx} u_{xx} - u^2 u_t u_{xxx} + 4\mathrm{e}^{2u} u^4 u_x^3 u_{xxx} - 6\mathrm{e}^{2u} u^3 u_{tx} u_x^2}{4u^4} \\
&+ \frac{6\mathrm{e}^{2u} u^2 u_t u_x^3 + 4\mathrm{e}^{2u} u^4 u_x u_{txx} + 8\mathrm{e}^{2u} u^4 u_{tx} u_{xx} - 3\mathrm{e}^{2u} u u_{tx} u_x^2 + 3\mathrm{e}^{2u} u^2 u_x u_{txx}}{4u^4} \\
&+ \frac{3\mathrm{e}^{2u} u^2 u_{tx} u_{xx} + \mathrm{e}^{2u} u^2 u_t u_{xxx} - 6\mathrm{e}^{2u} u u_t u_x^3 - 6\mathrm{e}^{2u} u^3 u_{txx} u_x - 6\mathrm{e}^{2u} u^3 u_{tx} u_{xx}}{4u^4} \\
&+ \frac{-2\mathrm{e}^{2u} u^3 u_t u_{xxx} + 6\mathrm{e}^{2u} u^2 u_{tx} u_x^2 + 6uu_x u_t u_{xx}}{4u^4}.
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= \frac{6t\mathrm{e}^{2u} u^2 u_x^2 + 6t\mathrm{e}^{2u} u^2 u_{xx} - 6t\mathrm{e}^{2u} u u_x^2 + 2x\mathrm{e}^{2u} u^2 - 3tu\mathrm{e}^{2u} u_{xx}}{12u^2} \\
&+ \frac{3t\mathrm{e}^{2u} u_x^2 - 2xu^2 + 3tuu_{xx} - 3tu_x^2}{12u^2},
\end{aligned}
$$

$$
\begin{aligned}
T_4^x &= \frac{2x\mathrm{e}^{2u} u^2 u_x^2 + 12t\mathrm{e}^{2u} u^2 u_x^2 u_{xx} - 3t\mathrm{e}^{2u} u u_{tx} + 6t\mathrm{e}^{2u} u^2 u_{tx} - 3tu_t u_x}{12u^2} \\
&+ \frac{3uu_x + 3tuu_{tx} + 3tu_t u_x \mathrm{e}^{2u} - 6tu\mathrm{e}^{2u} u_x u_t + 6t\mathrm{e}^{2u} u^2 u_{xx}^2}{12u^2} \\
&+ \frac{4\mathrm{e}^{2u} u^2 u_{xxx} + 6t\mathrm{e}^{2u} u_x^4 u^2 + 2\mathrm{e}^{2u} u^2 u_x - 3\mathrm{e}^{2u} u u_x}{12u^2}.
\end{aligned}
$$

$$
T_5^t = \frac{2\mathrm{e}^{2u} u^2 u_x^2 + 2\mathrm{e}^{2u} u^2 u_{xx} - 2\mathrm{e}^{2u} u u_x^2 - \mathrm{e}^{2u} u u_{xx} + \mathrm{e}^{2u} u_x^2 + uu_{xx} - u_x^2}{4u^2},
$$

$$
\begin{aligned}
T_5^x &= -\frac{2\mathrm{e}^{2u} u^2 u_x^4 + 4\mathrm{e}^{2u} u^2 u_x^2 u_{xx} + 2\mathrm{e}^{2u} u^2 u_{xx}^2 + 2\mathrm{e}^{2u} u^2 u_{tx}}{4u^2} \\
&+ \frac{-2\mathrm{e}^{2u} u u_t u_x - \mathrm{e}^{2u} u u_{tx} + \mathrm{e}^{2u} u_x u_t + uu_{tx} - u_x u_t}{4u^2}.
\end{aligned}
$$

$$
T_6^t = \frac{\mathrm{e}^{2u} - 1}{2},
$$

$$
T_6^x = -\frac{\mathrm{e}^{2u}}{2}\left(u_x^2 + 2u_{xx}\right).
$$

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