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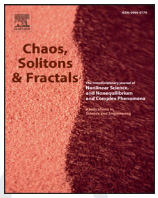

Group analysis, exact solutions and conservation laws of a generalized fifth order KdV equation

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ABSTRACT

We study the generalized fifth order KdV equation using group methods and conservation laws. All of the geometric vector fields of the special fifth order KdV equation are presented. By using the nonclassical Lie group method, it is show that this equation does not admit nonclassical type symmetries. Then, on the basis of the optimal system, the symmetry reductions and exact solutions to this equation are constructed. For some special cases, we obtain additional nontrivial conservation laws and scaling symmetries.

1. Introduction

Nonlinear evolution equations (NLEEs) are of importance in nonlinear science, in particular in applied mathematics and theoretical physics. Their solutions are important in the understanding of nonlinear interaction and behaviors of complex system. There are various techniques [1–17] used to deal with NLEEs, some of the commonly used ways involve the generalized symmetries, nonlocal symmetries, nonclassical Lie group and classical Lie group method.

It is well known that differential equations (DEs) have a number fundamental structures, that is, symmetries and conservation laws (CLs). CLs play a key roles in DEs analysis, particularly in studies of existence, uniqueness and stability of solutions. Various approaches have been used to handle symmetries and conservation laws of PDE systems (see [1–6] and the references therein).

In the present paper, we use the group method and the multiplier approach to study the fifth order KdV equation

$$u_t + \alpha u^n u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0. \quad (1)$$

In particular, for $n = 2$, one can get

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0, \quad (2)$$

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here α, β and γ are nonzero constants. Clearly, this equation has two dispersive terms u_{xxx} and u_{xxxxx}. By choosing the real values of the parameters n, α, β and γ, one can get a variety of fKdV equations [7,8] such as, when $n = 2$, the Sawada–Kotera (SK) equation

$$u_t + 5u^2u_x + 5u_xu_{xx} + 5uu_{xxx} + u_{xxxxx} = 0, \tag{3}$$

the Caudrey–Dodd–Gibbon (CDG) equation

$$u_t + 180u^2u_x + 30u_xu_{xx} + 30uu_{xxx} + u_{xxxxx} = 0, \tag{4}$$

the Lax equation

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{5}$$

the Kaup–Kupershmidt (KK) equation

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{6}$$

the Ito equation

$$u_t + 2u^2u_x + 6u_xu_{xx} + 3uu_{xxx} + u_{xxxxx} = 0. \tag{7}$$

Many authors have been studied these equations using different approaches. There is still, however, a lot of room for extensions and improvements. in particular exact solutions, symmetries and conservation law.

The main purpose of this paper is to investigate symmetry and conservation law classification of the generalized KdV equation. We will show that the particular case $n = 2$ is special as it is the only case that admit a scaling symmetry. The paper is organized as follows. In Section 2, all of the vector fields of Eq. (1) are constructed. In Section 3, we consider the special case of n, that is $n = 2$. In this case, the vector fields and some exact solutions are obtained. In Section 4, we study the soliton solutions of the equation. In Section 5, we find the conservation laws. Finally, conclusions and some remarks are given in Section 5.

2. Group analysis of the generalized fifth-order KdV equation

Consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2),\\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2),\\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{8}$$

with a small parameter $\epsilon \ll 1$, and the above group of transformations infinitesimal generator can read

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{9}$$

and we need to solve the coefficient functions $\tau(x, t, u)$, $\xi(x, t, u)$, $\eta(x, t, u)$.

Meanwhile, V must satisfy Lie's symmetry condition

$$pr^{(5)}V(\Delta)|_{\Delta=0} = 0, \tag{10}$$

where

$$\Delta = u_t + \alpha u^2u_x + \beta u_xu_{xx} + \gamma uu_{xxx} + u_{xxxxx}. \tag{11}$$

By using the fifth prolongation $Pr^{(5)}V$ to Eq. (1), one can see that the coefficient functions satisfy the following equation:

$$\begin{aligned} &\eta^t + n\alpha\eta u_x + \alpha u^n\eta^x + \beta\eta^x u_{xx} + \beta\eta^{xx}u_x\\ &+\gamma\eta u_{xxx} + \gamma\eta^{xxx}u + \eta^{xxxxx} = 0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \eta^t &= D_t(\eta) - u_xD_t(\xi) - u_tD_t(\tau),\\ \eta^x &= D_x(\eta) - u_xD_x(\xi) - u_tD_x(\tau),\\ &\cdots \end{aligned} \tag{13}$$

Here, D_i are the total derivative operators defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots \quad i = 1, 2, \tag{14}$$

and $(x^1, x^2) = (t, x)$.

On the basis of the Lie symmetry analysis method, one can get

$$\tau = c_1,\ \ \xi = c_2,\ \ \eta = 0, \tag{15}$$

where c_1 and c_2 are arbitrary constants. So one can have the geometric vector fields

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}. \tag{16}$$

3. Classical, nonclassical, potential symmetry and exact solutions for $n = 2$

In this section, we use the classical and nonclassical symmetry method to handle the fifth-order equation for $n = 2$.

3.1. Classical symmetry analysis

On the basis of the Lie symmetry analysis method, one can get

$$\tau = 5c_1t + c_2,\ \ \xi = c_1x + c_3,\ \ \eta = -2c_1u, \tag{17}$$

where c_1, c_2 and c_3 are arbitrary constants. So one can have the geometric vector fields

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial t},\ V_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{18}$$

3.2. Nonclassical symmetry analysis

In the previous subsection, we used the classical symmetry method to deal with the fifth order KdV equation. Next, we employ the nonclassical symmetry method [9,10] to study the fifth order KdV equation. The aim is that nonclassical symmetries are much more numerous than classical ones and maybe get more solutions of PDEs. In terms of the classical symmetry, the invariance surface condition should been added:

$$\Delta_1 = \eta - \xi u_x - \tau u_t. \tag{19}$$

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If the vector field (9) is a nonclassical symmetry of (1), which should satisfy

$$pr^{(5)}V(\Delta)|_{\Delta=0,\Delta_1=0} = 0. \tag{20}$$

Consider the nature of the invariant surface condition (19), without loss of generality, there are two cases to arise: (i)$\tau = 1$; (ii)$\tau = 0, \xi = 1$. More details see [9,10] and the references therein. In following, we will consider them respectively.

1. $\tau = 1$.

Then from the invariance surface condition (19), one can get

$$u_t = \eta - \xi u_x. \tag{21}$$

After differentiating (21) with respect to x, and then replacing u_t by using $\eta - \xi u_x$, one can get

$$\begin{aligned}
&\eta_t - \xi_t u_x + \eta_u(\eta - \xi u_x) + 2\alpha\eta u u_x + \alpha u^2(\eta_x + (\eta_u - \xi_x)u_x)\\
&+\beta u_{xx}(\eta_x + (\eta_u - \xi_x)u_x) + \beta u_x(\eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x\\
&+(\eta_{uu} - 2\xi_{xu}u_x^2) + (\eta_u - 2\xi_x u_{xx})) + \gamma\eta u_{xxx}\\
&+\gamma u[\eta_{xxx} + 3\eta_{xxu}u_x + 3\eta_{xuu}u_x^2 + 3\eta_{xu}u_{xx} + \eta_{uuu}u_x^3 + 3\eta_{uu}u_x u_{xx}\\
&+\eta_u u_{xxx} - \xi_{xxx}u_x - 3\xi_{xx}u_{xx} - 3\xi_x u_{xxx}] + 5\eta_{xxxxu}u_x + 10\eta_{xxxu}u_{xx}\\
&+10\eta_{xxu}u_{xxx} + 3\eta_{xu}u_{xxxx} + \eta_u u_{xxxxx} - \xi_{xxxxx}u_x - 5\xi_{xxxx}u_{xx}\\
&-10\xi_{xxx}u_{xxx} + 10\eta_{uu}u_{xx}u_{xxx} + \eta_{xxxxx} - 10\xi_{xx}u_{xxxx} - 5\xi_x u_{xxxxx}\\
&+10\eta_{xxuu}u_x^2 + 10\eta_{xxuuu}u_x^3 + 5\eta_{xuuuu}u_x^4 + 15\eta_{xuuu}u_{xx}^2 + \eta_{uuuuu}u_x^5\\
&+30\eta_{xuuu}u_x^2 u_{xx} + 20\eta_{xuu}u_x u_{xxx} + 30\eta_{xxuu}u_x u_{xx} + 5\eta_{uu}u_x u_{xxxx}\\
&+15\eta_{uuu}u_x u_{xx}^2 + 10\eta_{uuuu}u_x^3 u_{xx} + 10\eta_{uuu}u_x^2 u_{xxx} = 0.
\end{aligned} \tag{22}$$

Solving the overdetermined system of equations, leads to

$$\eta = \frac{-2u}{5t + C_1}, \xi = \frac{x + C_2}{5t + C_1}, \tag{23}$$

where C_1 and C_2 are arbitrary constants. Consequently, one can get the corresponding "nonclassical symmetry"is

$$V_4 = \frac{x + C_2}{5t + C_1}\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{-2u}{5t + C_1}\frac{\partial}{\partial u}. \tag{24}$$

It is cleat that, $V_4 = V_3 + C_1V_2 + C_2V_1$, it is the classical symmetry. Also, we can find the equation does not have nonclassical symmetries.

2. $\tau = 0,\ \xi = 1$.

Now, using the same approach as before, one can have

$$\eta = u_x. \tag{25}$$

Then, solving the overdetermined equations, one can get

$$\tau = 0,\ \ \xi = 1, \eta = 0. \tag{26}$$

That is to say, in this case, we could not get supplementary symmetries, of non-classical type. This also means no new explicit solutions can be constructed in the case of $\tau = 0$, $\xi = 1$.

3.3. Potential symmetry analysis

Suppose (1) can be written an a conservation law,

$$D_t T(x, t, u) + D_x X(x, t, u) = 0. \tag{27}$$

The PDE system $S(x, t, u, v) = 0$ given by

$$\begin{aligned}
&v_x = u,\\
&v_t = -\left(\frac{1}{3}\alpha u^3 + \frac{1}{2}(\beta - \gamma)u_x^2 + \gamma u u_{xx} + u_{xxxx}\right).
\end{aligned} \tag{28}$$

After repeating previous steps, one can get the coefficient functions $\tau(x, t, u, v), \xi(x, t, u, v), \eta(x, t, u, v)$ and $\psi(x, t, u, v)$ are:

$$\tau = c_1t + c_2,\ \ \xi = \frac{c_1}{3}x + c_4,\ \ \eta = -\frac{2c_1}{3}u,\ \psi = -c_1v + c_3. \tag{29}$$

One can find out that this equation does not have potential symmetry.

3.4. Symmetry reductions and group-invariant solutions for $n = 2$

In the previous section, we use the classical and nonclassical group method to deal with the fifth order KdV equation In this section, by using the optimal system, we give some group-invariant solutions.

3.4.1. One-dimensional optimal system of subalgebras

In order to get the optimal system, we applying the adjoint transformations formula [1] given by

$$Ad(exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{1}{2}\epsilon^2[V_i, [V_i, V_j]] - \cdots \tag{30}$$

where ϵ is a nonzero constant. Here $[V_i, V_j]$ is the commutator for the Lie algebra given by

$$[V_i, V_j] = V_iV_j - V_jV_i. \tag{31}$$

We can get an optimal system of one-dimensional subalgebras:

$$V_1, V_2 + \lambda V_1, V_3. \tag{32}$$

3.4.2. Symmetry reductions

In the present subsection, we employ the optimal system of one-dimensional subalgebras to deal with (1). and in the next subsection we will give some exact solutions of (1).

(1) $\boldsymbol{V_1}$.

For the generator V_1, one can get the group-invariant solution is $u = f(\xi)$, and $\xi = t$ is the group-invariant, in this case, one can get trivial solution $u(x, t) = C$, and C is a constant quantity.

(2) $V_2 + \lambda V_1$.

For the case of $V_2 + \lambda V_1$, we get the group-invariant solutions

$$u = f(\xi), \tag{33}$$

where $\xi = x - \lambda t$. Plugging (33) into (2), one can get the following ODE:

$$-\lambda f' + \alpha f^2 f' + \beta f' f'' + \gamma f f''' + f^{(5)} = 0. \tag{34}$$

In particular, if $\beta = 2\gamma$, and each term is multiplied by f, and integral once, one can get

$$-\frac{\lambda}{2}f^2 + \frac{1}{4}\alpha f^4 + \gamma f^2 f'' + f f^{(4)} - f' f''' + \frac{1}{2}(f'')^2 + k = 0, \tag{35}$$

where k is an integration constant.

(3) $\boldsymbol{V_3}$.

In the case of the generator V_3, we get

$$u = t^{-\frac{2}{5}} f(\xi), \xi = x t^{-\frac{1}{5}}. \tag{36}$$

Substitution of (36) into (2), one can lead to

$$-\frac{2}{5}f - \frac{1}{5}\xi f' + \alpha f^2 f' + \beta f' f'' + \gamma f f''' + f^{(5)} = 0. \tag{37}$$

3.5. Exact group-invariant solutions using power serious method

Supposing that (34) has the following solutions

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{38}$$

Substituting (38) into (36), one can have

$$\begin{aligned}
&120c_5 + \sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\xi^n \\
&+\alpha c_0^2 c_1 + \alpha \sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_j c_{k-j} c_{n+1-k}\xi^n \\
&+2\beta c_1 c_2 + \beta \sum_{n=1}^{\infty}\sum_{k=0}^{n}(k+1) \\
&\times (n+1-k)(n+2-k)c_{k+1}c_{n+2-k}\xi^n \\
&+6\gamma c_0 c_3 + \gamma \sum_{n=1}^{\infty}\sum_{k=0}^{n}(n+1-k)(n+2-k) \\
&\times (n+3-k)c_k c_{n+3-k}\xi^n \\
&-\frac{1}{5}\sum_{n=1}^{\infty} n c_n \xi^n - \left(\frac{2}{5}c_0 + \frac{2}{5}\sum_{n=1}^{\infty} c_n \xi^n\right) = 0.
\end{aligned} \tag{39}$$

Comparing coefficients for $n = 0$ in (39), one yields

$$c_5 = \frac{\frac{2}{5}c_0 - \alpha c_0^2 c_1 - 2\beta c_1 c_2 - 6\gamma c_0 c_3}{120}. \tag{40}$$

For general case, if $n \geq 1$, one can get

$$\begin{aligned}
c_{n+5} = &-\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \\
&\times \Bigg(\alpha \sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_j c_{k-j} c_{n+1-k} \\
&+\beta \sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \\
&+\gamma \sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)c_k c_{n+3-k} \\
&-\frac{1}{5}n c_n - \frac{2}{5}c_n\Bigg).
\end{aligned} \tag{41}$$

In this way, the power series solution of can be rewritten

$$\begin{aligned}
f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \sum_{n=1}^{\infty} c_{n+5}\xi^{n+5} \\
&= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 \\
&\quad + \frac{\frac{2}{5}c_0 - \alpha c_0^2 c_1 - 2\beta c_1 c_2 - 6\gamma c_0 c_3}{120}\xi^5 \\
&\quad - \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \\
&\quad \times \Bigg(\alpha \sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_j c_{k-j} c_{n+1-k} \\
&\quad +\beta \sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \\
&\quad +\gamma \sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)c_k c_{n+3-k} \\
&\quad -\frac{1}{5}n c_n - \frac{2}{5}c_n\Bigg)\xi^{n+5}.
\end{aligned} \tag{42}$$

Therefore, one can get

$$\begin{aligned}
u(x,t) &= \Bigg[c_0 + c_1(xt^{-\frac{1}{5}}) + c_2(xt^{-\frac{1}{5}})^2 + c_3(xt^{-\frac{1}{5}})^3 \\
&\quad + c_4(xt^{-\frac{1}{5}})^4 + \sum_{n=0}^{\infty} c_{n+5}(xt^{-\frac{1}{5}})^{n+5}\Bigg](t^{-\frac{2}{5}}) \\
&= \Bigg[c_0 + c_1(xt^{-\frac{1}{5}}) + c_2(xt^{-\frac{1}{5}})^2 + c_3(xt^{-\frac{1}{5}})^3 \\
&\quad + c_4(xt^{-\frac{1}{5}})^4 \\
&\quad - \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \\
&\quad \times \Bigg(\alpha \sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_j c_{k-j} c_{n+1-k} \\
&\quad +\beta \sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \\
&\quad +\gamma \sum_{k=0}^{n}(n+1-k)(n+2-k)(n+3-k)c_k c_{n+3-k} \\
&\quad -\frac{1}{5}n c_n - \frac{2}{5}c_n\Bigg)(xt^{-\frac{1}{5}})^{n+5}\Bigg](t^{-\frac{2}{5}}),
\end{aligned} \tag{43}$$

here $c_i (i = 0, 1, 2, 3, 4)$ are arbitrary constants.

Remark. The exact solution of (43) also can be fixed in similarly way. The details are omitted here. Q4

In addition, by using the Maple soft, one can get following Jacobi function and Weierstrass elliptic function solutions:

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$$u(x,t) = \frac{-6C_3^2 SN\left(-6\frac{\left(2\beta\gamma+\beta^2-\sqrt{\beta^4+4\beta^3\gamma+4\beta^2\gamma^2-40\alpha\beta^2}-12\alpha\right)C_3{}^5 t}{\alpha}+C_3x+C_2, i\right)^2}{\beta\left(12\,\gamma^2+12\,\beta\,\gamma+6\,\beta^2+6\,\sqrt{\beta^4+4\,\beta^3\gamma+4\,\beta^2\gamma^2-40\,\alpha\,\beta^2}-120\,\alpha\right)}$$
$$\times\frac{\frac{-6\gamma^2\left(2\beta\gamma+\beta^2-\sqrt{\beta^4+4\beta^3\gamma+4\beta^2\gamma^2-40\alpha\beta^2}-12\alpha\right)}{\alpha}-72\gamma^2+120\beta\gamma}{\beta\left(12\,\gamma^2+12\,\beta\,\gamma+6\,\beta^2+6\,\sqrt{\beta^4+4\,\beta^3\gamma+4\,\beta^2\gamma^2-40\,\alpha\,\beta^2}-120\,\alpha\right)}$$
$$\times\frac{-60\beta^2-60\sqrt{\beta^4+4\beta^3\gamma+4\beta^2\gamma^2-40\alpha\beta^2}}{\beta\left(12\,\gamma^2+12\,\beta\,\gamma+6\,\beta^2+6\,\sqrt{\beta^4+4\,\beta^3\gamma+4\,\beta^2\gamma^2-40\,\alpha\,\beta^2}-120\,\alpha\right)}. \quad (44)$$

$$u(x,t) = \frac{-6C_3^2 SN\left(-6\frac{\left(2\beta\gamma+\beta^2+\sqrt{\beta^4+4\beta^3\gamma+4\beta^2\gamma^2-40\alpha\beta^2}-12\alpha\right)C_3{}^5 t}{\alpha}+C_3x+C_2, i\right)^2}{\beta\left(12\,\gamma^2+12\,\beta\,\gamma+6\,\beta^2+6\,\sqrt{\beta^4+4\,\beta^3\gamma+4\,\beta^2\gamma^2-40\,\alpha\,\beta^2}-120\,\alpha\right)}$$
$$\times\frac{\frac{-6\gamma^2\left(2\beta\gamma+\beta^2+\sqrt{\beta^4+4\beta^3\gamma+4\beta^2\gamma^2-40\alpha\beta^2}-12\alpha\right)}{\alpha}-72\gamma^2+120\beta\gamma}{\beta\left(12\,\gamma^2+12\,\beta\,\gamma+6\,\beta^2+6\,\sqrt{\beta^4+4\,\beta^3\gamma+4\,\beta^2\gamma^2-40\,\alpha\,\beta^2}-120\,\alpha\right)}$$
$$\times\frac{-60\beta^2+60\sqrt{\beta^4+4\beta^3\gamma+4\beta^2\gamma^2-40\alpha\beta^2}}{\beta\left(12\,\gamma^2+12\beta\gamma+6\beta^2-6\sqrt{\beta^4+4\,\beta^3\gamma+4\,\beta^2\gamma^2-40\,\alpha\,\beta^2}-120\,\alpha\right)}. \quad (45)$$

$$u(x,t) = -3\wp\left(\frac{1}{2}C_4{}^5C_2(-3M+36)t+C_4x+C_3, C_2, C_1\right) \times\left(2\gamma+\beta-\sqrt{\beta^2+4\beta\gamma+4\gamma^2-40\alpha}\right)C_4{}^2\alpha^{-1}. \quad (46)$$

$$u(x,t) = -3\wp\left(\frac{1}{2}C_4{}^5C_2(-3N+36)t+C_4x+C_3, C_2, C_1\right) \times\left(2\gamma+\beta+\sqrt{\beta^2+4\beta\gamma+4\gamma^2-40\alpha}\right)C_4{}^2\alpha^{-1}. \quad (47)$$

4. Solitons solutions

This section will obtain solitary wave solutions to the model equation given by (1). The method of undetermined coefficients will be adopted to retrieve these solitons. In this case we rewrite Eq. (1) as follows:

$$u_t+\alpha u^n u_x+\beta u_x u_{xx}+\gamma u u_{xxx}+\delta u_{xxxxx}=0. \quad (48)$$

There are four types of solutions that are going to be extracted for (48) with the aid of the method of undetermined coefficients. They are in the next few subsections:

4.1. Solitary waves

In order to obtain solitary waves, the starting hypothesis is:

$$u(x,t)=A\operatorname{sech}^p\tau \quad (49)$$

with

$$\tau=B(x-vt) \quad (50)$$

where A is the amplitude of the soliton, $p>0$ is a parameter that will be obtained with the aid of the balancing principle, B represents the inverse width of the soliton while v is the speed. By substituting (49) into (48) one obtain

$$\begin{aligned}&(v-p^4\delta B^4)\operatorname{sech}^{p+1}\tau-p^2(\beta+\gamma)AB^2\operatorname{sech}^{2p+1}\tau\\&-\alpha A^n\operatorname{sech}^{(n+1)p+1}\tau\\&+(1+p)(2\gamma+p(\beta+\gamma))AB^2\operatorname{sech}^{2p+3}\tau\\&+2(1+p)(2+p)[2+p(2+p)]\delta B^4\operatorname{sech}^{p+3}\tau\\&-(1+p)(2+p)(3+p)(4+p)\delta B^4\operatorname{sech}^{p+5}\tau=0\end{aligned} \quad (51)$$

The balancing principle allows to equate the exponents $(n+1)p+1$ with $p+5$ from which

$$p=\frac{4}{n}. \quad (52)$$

Notice also that the same principle allows to equate $2p+3$ with $p+5$, and $2p+1$ with $p+3$, both situations leading to

$$p=2 \quad (53)$$

Thus, from (52) and (53) we have

$$n=2. \quad (54)$$

Consequently, the system (51) collapses into

$$\begin{aligned}&(v-16\delta B^4)\operatorname{sech}^3\tau-4[(\beta+\gamma)A-60\delta B^2]B^2\operatorname{sech}^5\tau\\&-[360\delta B^4+\alpha A^2-6(2\gamma+\beta)AB^2]\operatorname{sech}^7\tau=0.\end{aligned} \quad (55)$$

After equating the coefficients of the linearly independent functions $\operatorname{sech}^j\tau$ for $j=3,5,7$ to zero one get

$$v=16\delta B^4, \quad (56)$$

$$A = \frac{60\delta B^2}{\beta+\gamma} \tag{57}$$

constrained by

$$\beta+\gamma \neq 0, \tag{58}$$

and the identity

$$(2\gamma+\beta)(\gamma+\beta)-(\beta+\gamma)^2-10\alpha\delta=0. \tag{59}$$

Therefore, the solitary wave solution to (48) is given by

$$u(x,t)=A\,\mathrm{sech}^2[B(x-vt)] \tag{60}$$

where the speed v is given in (56), while the amplitude A is provided in (57). For the solution to exist the identity (59) has to be satisfied along with the condition (58).

4.2. Shock waves

In order to solve the generalized fifth order KdV Eq. (48) for shock wave the starting hypothesis is taken to be

$$u(x,t)=A\tanh^p\tau \tag{61}$$

with τ as defined in (50), A and B are free parameters, and $p>0$ is a key parameter to be determined. Substituting Eq. (49) into Eq. (48) gives in a simplified form

$$\begin{aligned}&[v-2p^2(5+3p^2)\delta B^4]\tanh^{p-1}\tau-\alpha A^n\tanh^{(n+1)p-1}\tau\\&+4(p-1)(p-2)[2+p(p-2)]\delta B^4\tanh^{p-3}\tau\\&-(p-1)(p-2)(p-3)(p-4)\delta B^4\tanh^{p-5}\tau\\&+2p^2(\beta+\gamma)AB^2\tanh^{2p-1}\tau+(p-1)\\&\times[2\gamma-p(\beta+\gamma)]AB^2\tanh^{2p-3}\tau\\&+4(p+1)(p+2)[2+p(2+p)]\delta B^4\tanh^{p+1}\tau\\&-(p+1)(p+2)(p+3)(p+4)\delta B^4\tanh^{p+3}\tau\\&-(p+1)[2\gamma+p(\beta+\gamma)]AB^2\tanh^{2p+1}\tau=0.\end{aligned} \tag{62}$$

By the balancing principle, equating the exponents $(n+1)p-1$ and $p+3$ leads to (52), but also it is possible to equate $2p+1$ with $p+3$ resulting in (53) and consequently one get (54). In view of this values of p and n, the Eq. (62) can be rewritten as

$$\begin{aligned}&[v-136\delta B^4-2\beta AB^2]\tanh\tau+8[(\beta+\gamma)A\\&\quad+60\delta B^2]B^2\tanh^3\tau-[6(\beta+2\gamma)AB^2\\&\quad+360\delta B^4+\alpha A^2]\tanh^5\tau=0.\end{aligned} \tag{63}$$

Equating to zero the coefficients of the linearly independent functions $\tanh^j\tau$ for $j=1,3,5$ lead us to

$$v=2(\beta A+68\delta B^2)B^2, \tag{64}$$

and the amplitude becomes

$$A=-\frac{60\delta B^2}{\beta+\gamma}. \tag{65}$$

along with the condition (58). In addition, the identity

$$(\beta+\gamma)^2-(2\gamma+\beta)(\gamma+\beta)+10\alpha\delta=0. \tag{66}$$

has to be satisfied in order for the shock waves to exist. Therefore, the shock wave to the generalized fifth order KdV Eq. (48) is given by

$$u(x,t)=A\tanh^2[B(x-vt)] \tag{67}$$

where the amplitude is given in (65) while the speed turns to be (64) along with the corresponding conditions. In addition, the condition (66) has to be satisfied.

4.3. Singular solitary waves (Type-I)

For type-I singular solitary wave solution to Eq. (48) the starting hypothesis is

$$u(x,t)=A\,\mathrm{csch}^p\tau \tag{68}$$

where the parameter $p>0$ will be determined with the help of the balancing principle, A and B are free parameters, while τ retain the same meaning as in (50). The substitution of (68) into (48) leads to

$$\begin{aligned}&(v-p^4\delta B^4)\mathrm{csch}^{p+1}\tau-p^2(\beta+\gamma)AB^2\mathrm{csch}^{2p+1}\tau\\&-\alpha A^n\mathrm{csch}^{(n+1)p+1}\tau-2(1+p)(2+p)\\&\times(2+p(2+p))\delta B^4\mathrm{csch}^{p+3}\tau\\&-(1+p)[2\gamma+p(\beta+\gamma)]AB^2\mathrm{csch}^{2p+3}\tau\\&-(1+p)(2+p)(3+p)(4+p)\delta B^4\mathrm{csch}^{p+5}\tau=0.\end{aligned} \tag{69}$$

As in Eq. (51), it is possible to equate $(n+1)p+1$ with $p+5$, $2p+3$ with $p+5$, and also $2p+1$ with $p+3$, thus leading to (53), and consequently (54). As a consequence, the Eq. (69) becomes

$$\begin{aligned}&(v-16\delta B^4)\mathrm{csch}^3\tau-4[(\beta+\gamma)A+60\delta B^2]B^2\mathrm{csch}^5\tau\\&-[360\delta B^4+\alpha A^2+6(2\gamma+\beta)AB^2]\mathrm{csch}^7\tau=0.\end{aligned} \tag{70}$$

After equating the coefficients of the linearly independent functions $\mathrm{csch}^j\tau$ for $j=3,5,7$ to zero one get the wave speed as in (56), the amplitude as in (65), and the identity turns to be the same as in (66). Thus, the type-I singular solitary wave solution to the generalized KdV Eq. (48) is given by

$$u(x,t)=A\,\mathrm{csch}^2[B(x-vt)] \tag{71}$$

where the soliton speed and amplitude are given in (56) and (65) respectively, while the identity (66) has to be satisfied in order to preserve the existence of the solitary wave.

4.4. Singular solitary waves (Type-II)

To obtain type-II singular solitary wave solutions to Eq. (48) the starting Ansatz is

$$u(x,t)=A\coth^p\tau \tag{72}$$

where the meaning of A, p and τ are as in the previous subsections. By inserting (72) into the fifth order KdV Eq. (48) one get

$$\begin{aligned}&[v-2p^2(5+3p^2)\delta B^4]\coth^{p-1}\tau-\alpha A^n\coth^{(n+1)p-1}\tau\\&+4(p-1)(p-2)[2+p(p-2)]\delta B^4\coth^{p-3}\tau\\&-(p-1)(p-2)(p-3)(p-4)\delta B^4\coth^{p-5}\tau\\&+2p^2(\beta+\gamma)AB^2\coth^{2p-1}\tau+(p-1)\\&\times[2\gamma-p(\beta+\gamma)]AB^2\coth^{2p-3}\tau\\&+4(p+1)(p+2)[2+p(2+p)]\delta B^4\coth^{p+1}\tau\\&-(p+1)(p+2)(p+3)(p+4)\delta B^4\coth^{p+3}\tau\\&-(p+1)[2\gamma+p(\beta+\gamma)]AB^2\coth^{2p+1}\tau=0.\end{aligned} \tag{73}$$

The balancing principle leads to (53) and (54). Both values reduce the Eq. (73) into

$$[\nu - 136\delta B^4 - 2\beta AB^2]\coth\tau + 8[(\beta+\gamma)A + 60\delta B^2]B^2\coth^3\tau - [6(\beta+2\gamma)AB^2 + 360\delta B^4 + \alpha A^2]\coth^5\tau = 0. \tag{74}$$

and consequently, the results (64)–(66) reappear. Finally, the type-II singular solitary wave solution to the generalized fifth order KdV Eq. (48) is given by

$$u(x,t) = A\coth^2[B(x-\nu t)] \tag{75}$$

where the amplitude is provided in (65) while the speed turns to be (64) along with the corresponding conditions.

5. Conservation laws

In this section, we employ the multipliers method [3–5] to deal with the conservation law. Firstly, we introduces some basic definitions and concepts.

From what has been described above, suppose the conservation law is given by $D_xT^x + D_tT^t =$ on the solutions of (1).

In Section 2, we got the Lie point symmetry generators $X_1 = \partial_t$ and $X_2 = \partial_x$. For the case $n = 2$, we obtain the additional scaling symmetry $X_3 = x\partial_x + 5t\partial_t - 2u\partial_u$.

In general, we have the only conserved vector based on the multiplier $Q = 1$ given by

$$T^x = \frac{1}{n+1}\alpha u^{n+1} + \frac{1}{2}(\beta-\gamma){u_x}^2 + \gamma u u_{xx} + u_{xxxx},$$
$$T^t = u. \tag{76}$$

In particular, if $\beta = 2\gamma$, one can get the multiplier $Q = u$, and get the conserved vector

$$T^x = \frac{1}{n+2}\alpha u^{n+2} + \gamma u^2 u_{xx} + u u_{xxxx} - u_x u_{xxx} + \frac{1}{2}u_{xx}^2,$$
$$T^t = \frac{1}{2}u^2. \tag{77}$$

For $n = 2$, subject to the condition

$$10\alpha + 2\beta^2 - 7\beta\gamma + 3\gamma^2 = 0, \tag{78}$$

we obtain an additional, second order, multiplier $\mathbf{Q} = \frac{1}{10}((2\beta-\gamma)u^2 + 10u_{xx})$ leading to the nontrivial conserved flow

$$\begin{aligned} T^x = {} & \frac{1}{200}(\alpha(8\beta-4\gamma)u^5 + 20(2\beta-\gamma)\gamma u^3 u_{xx} \\ & -20u(5u_{xt} - 2((\beta+2\gamma){u_{xx}}^2 + (-2\beta+\gamma)u_x u_{xxx})) \\ & -5u^2((10\alpha+2\beta^2-7\beta\gamma+3\gamma^2){u_x}^2 \\ & +4(-2\beta+\gamma)u_{xxxx}) + 20(5u_t u_x + 2(2\beta-\gamma){u_x}^2 u_{xx} \\ & -5({u_{xxx}}^2 - 2u_{xx}u_{xxxx}))), \\ T^t = {} & \frac{1}{30}((2\beta-\gamma)u^3 + 15uu_{xx}). \end{aligned} \tag{79}$$

We note that the action of X_3 on $\mathbf{Q}$ satisfies $X_3\mathbf{Q} = -4\mathbf{Q}$. For any solution $u(x, t)$, if u and its derivatives converge to $x \longrightarrow \pm\infty$, $\int_{-\infty}^{\infty} T^t dx$ provide conserved quantities.

From the solitary wave solution derived in the previous section, the conserved quantities, with $\delta = 1$, are as follows:

$$I_1 = \int_{-\infty}^{\infty} T^t dx = \int_{-\infty}^{\infty} u dx = \frac{2A}{B} \tag{80}$$

$$I_2 = \int_{-\infty}^{\infty} T^t dx = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx = \frac{2A^2}{3B} \tag{81}$$

$$I_3 = \int_{-\infty}^{\infty} T^t dx = \frac{1}{30}\int_{-\infty}^{\infty}\{(2\beta-\gamma)u^3 + 15uu_{xx}\}dx = \frac{8A^2}{225B}\{(2\beta-\gamma)A + 15B^2\} \tag{82}$$

Remark. It does not exist other nth order multiplier, in other words, it only exist, for the general case, second order multiplier.

6. Conclusion

In the present paper, using the group methods, and the multiplier approach, the generalized fifth-order KdV equation is studied. Furthermore, we derive the corresponding Lie algebra and the similarity reductions of special case of generalized fifth-order KdV equation. Also, we found that the analyzed model does not admit nonclassical type symmetries for $n = 2$. In addition, on the basis of symmetries, the optimal system is constructed, based on the optimal system, some exact solutions are presented. Meanwhile, some soliton solutions are presented. Finally, conservations laws are derived. These results are important for the understanding of nonlinear interaction and behaviors of complex system in some piratical physical problems.

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References

[1] Olver PJ. Application of Lie group to differential equation. New York: Springer; 1986.
[2] Ovsiannikov LV. Group analysis of differential equations. New York: Academic Press; 1982.
[3] Bluman GW, Kumei S. Symmetries and differential equations. New York: Springer; 1989.
[4] Bluman GW, Cheviakov A, Anco S. Applications of symmetry methods to partial differential equations. New York: Springer; 2010.
[5] Bluman GW, Anco S. Symmetry and integration methods for differential equations. New York: Springer; 2002.
[6] Ibragimov N.H., editor. CRC handbook of Lie group analysis of differential equations; vol. 1–3. Boca Raton, FL: CRC Press; 1994.
[7] Wazwaz AM. Soliton solutions for the fifth-order KdV equation and the Kawahara equation with time-dependent coefficients. Phys Scr 2010;82:035009.
[8] Liu HZ. Complete group classifications and symmetry reductions of the fractional fifth-order KdV types of equations. Stud Appl Math 2013;131:317–30.
[9] Chaolu T, Bluman GW. An algorithmic method for showing existence of nontrivial nonclassical symmetries of partial differential equations without solving determining equations. J Math Anal Appl 2014;411:281–96.

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[10] Bluman GW, Chaolu T. Conservation laws for nonlinear telegraph equations. J Math Anal Appl 2005;310:459–76.
[11] Anco SC, Bluman GW. Direct construction of conservation laws from field equations. Phys Rev Lett 1997;78:2869–73.
Q6
[12] Hirota R.. The direct method in soliton theoryCambridge:Cambridge University Press; 2004.
[13] Wang GW. Symmetry analysis and rogue wave solutions for the (2+1)-dimensional nonlinear Schrödinger equation with variable coefficients. Appl Math Lett 2016;56:56–64.
[14] Wang GW, Kara AH. Nonlocal symmetry analysis, explicit solutions and conservation laws for the fourth-order burgers' equation. Chaos Solitons Fractals 2015;81:290–8.
[15] Razborova P, Kara AH, Biswas A. Additional conservation laws for Rosenau-KdV-RLW equation with power law nonlinearity by lie symmetry. Nonlinear Dyn 2015;79:743–8.
[16] Wang GW, Fakhar K. Lie symmetry analysis, nonlinear self-adjointness and conservation laws to an extended (2+1)-dimensional Zakharov–Kuznetsov–Burgers equation. Comput. Fluids. 2015;119:143–8.
[17] Kara AH. On the reduction of some dispersionless integrable systems. Acta Appl Math 2014;132:371–6.

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